MATH 1650: SECTION 1.1: FUNCTIONS AND THEIR REPRESENTATIONS

DEFINITION: Given two sets A and B, a **function** from A to B is a process by which each element of A is matched with (or 'mapped to') one and only one element of B. We think of the set A as being the set of **inputs** to the function while the set B is the set of **outputs** from the function.

EXAMPLES:

- Suppose we look at the process 'L' which matches each student in the class with the first letter of their first name. Is L a function? To determine if L is a function, we need to first identify the inputs, outputs, and make sure that each input is matched to only one output.

  The inputs to L are (choose one): (students in the class) or (letters of the alphabet)

  The outputs from L are (choose one): (students in the class) or (letters of the alphabet)

  Hence, to see if the process L is a function, we need to make sure that each: (student in the class) or (letter of the alphabet) is matched with one and only one: (student in the class) or (letter of the alphabet)

  So, is L a function?

- Suppose we reverse the process 'L' from the first example and create the process 'S' which matches each letter of the alphabet with the student in the class whose first name begins with that letter. Is S a function? As usual, let's first determine the inputs and outputs.

  The inputs to S are (choose one): (students in the class) or (letters of the alphabet)

  The outputs from S are (choose one): (students in the class) or (letters of the alphabet)

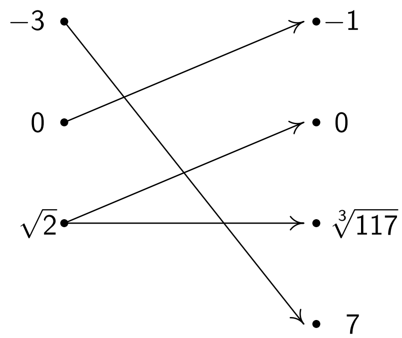
  Hence, to see if the process S is a function, we need to make sure that each: (student in the class) or (letter of the alphabet) is matched with one and only one: (student in the class) or (letter of the alphabet)

  So, is S a function?

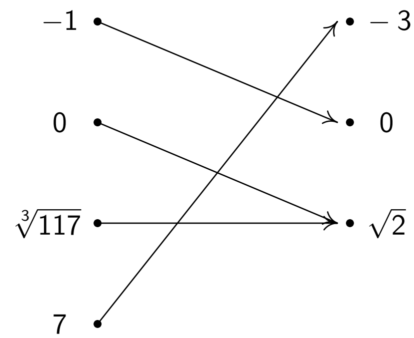
- Consider the process T that maps the time on a particular day to the temperature as registered by the Burke Lakefront Airport. Is T is a function? What about the reverse process?

Another way to describe functions is by visualizing them using ‘Mapping Diagrams’ as shown below. Here, the **inputs** to the mapping are shown on the **left** and they are matched to their corresponding **outputs** on the **right**.

- The mapping f:



- The mapping g:



- On the mapping f, the input -3 is matched to what output?
- On the mapping g, the input 0 is matched to what output?

Remember, to be a **function**, a mapping must match each input to **only one** output.

- Is the mapping f a function? Why or why not?

- Is the mapping g a function? Why or why not?

DEFINITION:

- The **domain** of a function is the set of all inputs to the function.
- The **range** of a function is the set of all outputs from the function.
- If the range is a subset of the real numbers:
  - The **minimum** of a function is the smallest number in the range (if it exists.)
  - The **maximum** of a function is the largest number in the range (if it exists.)

EXAMPLE:

- State the domain and range of the function g.

- Find the maximum and minimum of g.

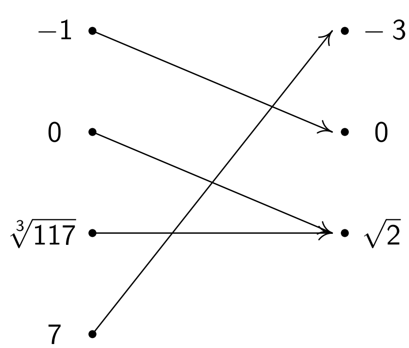
FUNCTION NOTATION

DEFINITION: Suppose the process f is a function and x is in the domain of f. The notation '$f(x)$' which is read 'f **of** x' represents the **output** from the process f using the **input** x. That is:

$$f(\text{input}) = \text{output}$$

EXAMPLE:

- Let L be the process by which each student is matched with the first letter of their first name. Suppose a student named Emmy Noether is in the class. What is L(Emmy Noether)?
- Consider the function g below:



Find and simplify the following:

1. $g(0)$
2. $g(7)$
3. $g(0) + g(7)$
4. $g(0 + 7)$
5. Solve $g(x) = 0$
6. Solve $g(t) = \sqrt{2}$

- Suppose T is the function which maps the time on January 1, 2019 to the temperature at Burke Lakefront Airport. More specifically, let $T(t)$ denote the temperature (in degrees Fahrenheit) t hours after 6 AM.

1. What does $T(0) = 45$ mean?
2. At noon, the temperature was recorded as 36°F. Express this relationship using function notation.
3. What do the solutions to $T(t) = 40$ represent?
4. What would be the significance of the minimum and maximum of T?

ALGEBRAIC REPRESENTATIONS OF FUNCTIONS

EXAMPLE:

- Let F be the process which takes a real number and performs the following sequence of operations:
  - Step 1: square the number
  - Step 2: add 1 to the result of Step 1.

  1. Find $F(3)$.
  2. Find $F(-2)$.
  3. Find a formula for $F(x)$.
  4. Solve $F(x) = 4$.

- Let us define the function g by the formula: $g(t) = 2t^2 - 3t + 1$. This means g takes the input 't' and returns the value of the expression '$2t^2 - 3t + 1$' as the output. For example, to find $g(-2)$, we substitute $t = -2$ into $2t^2 - 3t + 1$: $g(-2) = 2(-2)^2 - 3(-2) + 1 = 8 + 6 + 1 = 15$. Find and simplify the following:

  1. $g(0)$
  2. $g(-1)$
  3. $g(3a)$
  4. $3g(a)$
  5. $g(x + 2)$
  6. $g(x) + 2$
  7. $g(x) + g(2)$

DEFINITION: Given an equation involving two variables (say x and y) we say the equation describes 'y as a function of x' if each choice of x produces (at most) one resulting value of y.

EXAMPLE:

- Determine if the equation $x^3 - y^2 = 1$ describes y as a function of x.

  To determine if 'y' is a function of 'x,' we need to determine if

  each (choose one): x or y

  produces only one (choose one): x or y.

  To help us answer this question, we solve for y in terms of x:

$$\begin{aligned} x^3 - y^2 &= 1 \\ -y^2 &= 1 - x^3 \\ y^2 &= x^3 - 1 \\ y &= \pm\sqrt{x^3 - 1} \end{aligned}$$

  Is y a function of x? Explain.

- Determine if the equation $xy^3 = 8$ describes y as a function of x.

DEFINITION: If an equation describes y as a function of x, then there is a function f so that $y = f(x)$. In this case we say y is the **dependent** variable and x is the **independent** variable.

QUESTION: Which of the equations above describe x as a function of y?

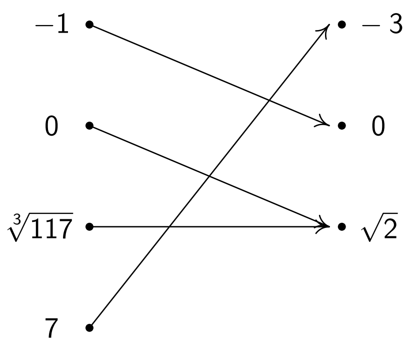
GEOMETRIC REPRESENTATIONS OF FUNCTIONS

Consider the function g described by the mapping diagram below on the left.
We can more succinctly describe g as a set of *ordered pairs* of the form (input, output):

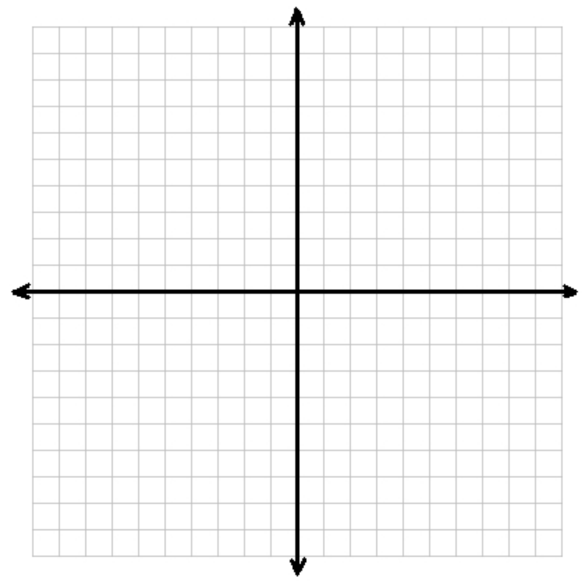
$$g = \{(-1, 0), (0, \sqrt{2}), (\sqrt[3]{117}, \sqrt{2}), (7, -3)\}$$

Plotting these points results in the **graph** of g. Graph g below on the right on the xy-plane.

A mapping diagram of g



The graph of g



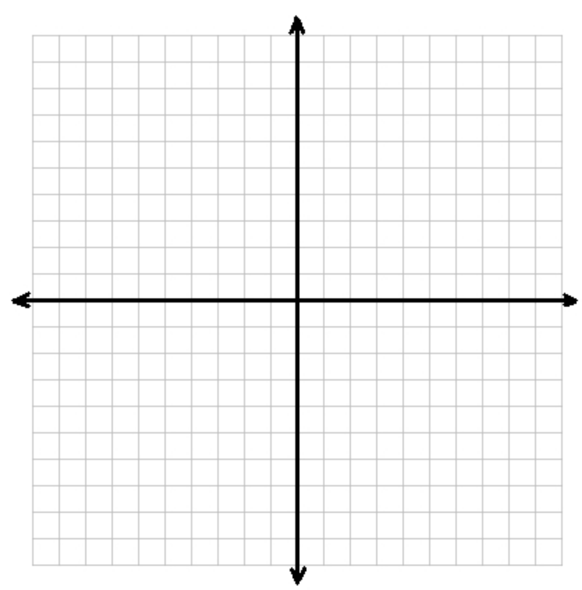
The graph of $y = g(x)$.

In general, given a function f, we may describe f as a set of ordered pairs:

$$\{(x, f(x)) \,|\, \text{for } x \text{ in the domain of } f\} = \{(\text{input}, \text{output})\}$$

EXAMPLE: Consider the set of points $F = \{(0, 3), (1, 0), (2, 3), (3, 0), (4, 3), (5, 0), (6, 3), \ldots\}$ in the xy-plane.

1. Explain why F represents y as a function of x.
2. Find $F(0)$.
3. Solve $F(x) = 0$.
4. State the domain and range of F.
5. Find the maximum and minimum of F.
6. Graph F using the axes provided on the right.



The graph of $y = F(x)$.

EXAMPLE: Which of the following sets of points in the xy-plane represent y as a function of x?

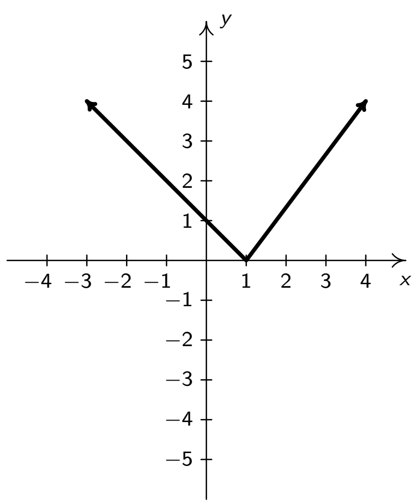
HINT: Try to find an equation that relates the x and y coordinates.

1. $h = \{(a, b) \mid b = |a|\}$

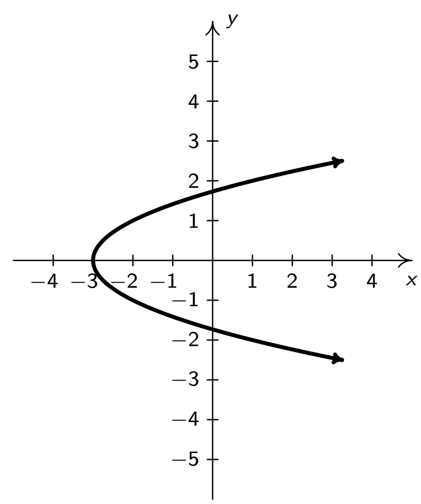
2. $G = \{\left(2t^2, t\right) \mid t \text{ is a real number}\}$

EXAMPLE: Consider the sets of points S and T plotted below on the xy-plane.

- Graph of the set S.



- Graph of the set T.



1. Does the set S represent y as a function of x?
   HINT: Remember, this means for each 'x' there must be only one 'y'.

2. Does the set T represent y as a function of x?
   HINT: Remember, this means for each 'x' there must be only one 'y'.

For sets of points described graphically, we have a relatively quick way to determine if they represent functions:

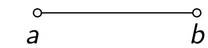
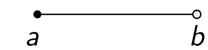
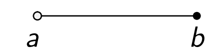
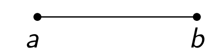
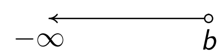
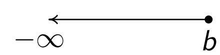
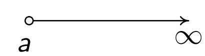
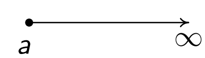
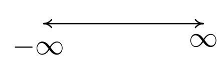
The Vertical Line Test' (VLT): A graph on the xy-plane represents y as a function of x if and only if no vertical line intersects the graph more than once. If the latter is true, we say the relation **passes** the Vertical Line Test.

EXAMPLE: Apply the VLT to the graphs above.

QUESTION: How would we determine if a graph on the xy-plane represents x as a function of y?

REVIEW OF INTERVAL NOTATION

Let a and b be real numbers with $a < b$.

Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leq x < b\}$	$[a, b)$	
$\{x \mid a < x \leq b\}$	$(a, b]$	
$\{x \mid a \leq x \leq b\}$	$[a, b]$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid x \leq b\}$	$(-\infty, b]$	
$\{x \mid x > a\}$	(a, ∞)	
$\{x \mid x \geq a\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

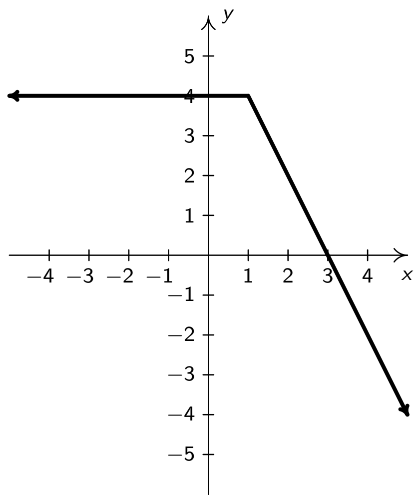
Notational Rules:

- It's always the smaller number (or $-\infty$) first.
- If a number is not included, we use a parenthesis: '(' or ')' ('∞' and '$-\infty$' are never included!).
- If the number is included, we use a bracket: '[' or ']'.

NOTE: When applicable, we will be using interval notation, when describing sets of real numbers.

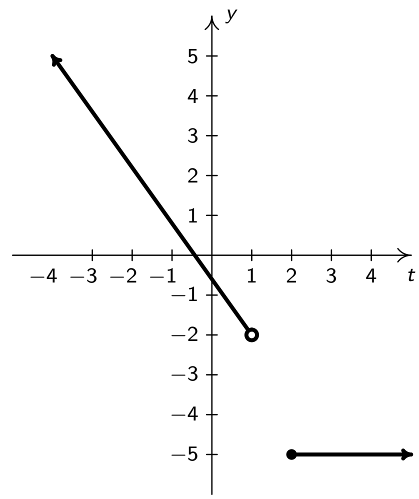
EXAMPLE: Use the graphs of the functions below to answer the indicated questions.

HINT: Remember that the points on the graph of a function are of the form: (input, output).



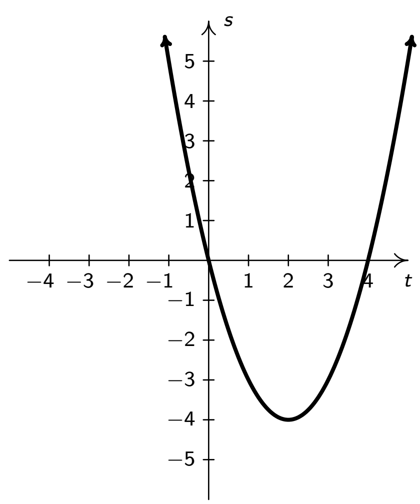
$y = C(x)$

- Domain:
- Range:
- Minimum:
- Maximum:
- Find $C(-1.1)$.



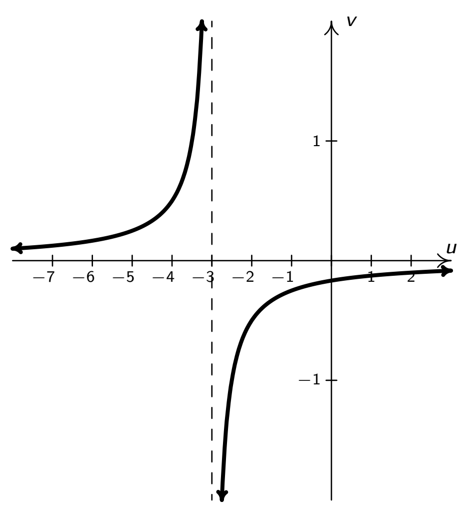
$y = N(t)$

- Domain:
- Range:
- Minimum:
- Maximum:
- Find $N(2)$.



$s = h(t)$

- Domain:
- Range:
- Minimum:
- Maximum:
- Solve $h(t) = 0$.



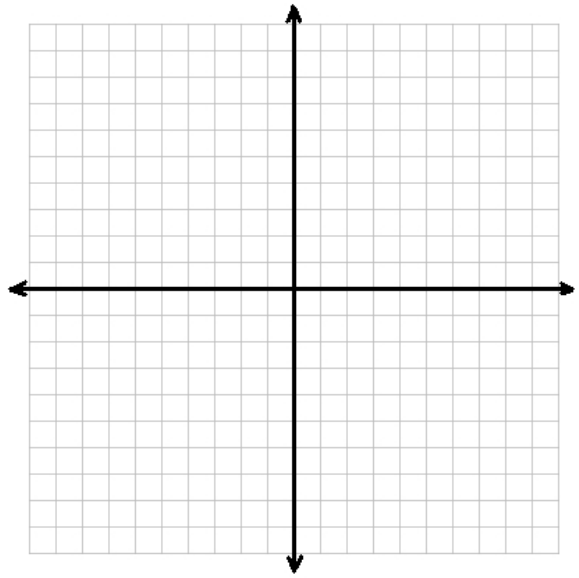
$v = r(u)$

- Domain:
- Range:
- Minimum:
- Maximum:
- Solve $r(u) > 0$.

EXAMPLE: Suppose we wish to graph $f(x) = 12 - 4x - x^2$ on the xy-plane. We can create a table of **sample** points using the given $x-$values to start. We can then refine our graph using desmos or a graphing utility.[1]

x	$y = f(x)$	$(x, f(x))$
-3		
-2		
-1		
0		
1		
2		
3		

The graph of $y = f(x) = 12 - 4x - x^2$



From the graph, determine the:

- domain:
- range:
- maximum:
- minimum:

EXAMPLE: Suppose $T(t)$ represents the temperature in degrees Fahrenheit of piping hot coffee t minutes after it is served.

1. Explain why T is a function.
2. What does $T(15)$ represent?
3. What do the solutions to $T(t) = 120$ represent?
4. What would the domain of T be?
5. What information would you have to know to in order to determine the range of T?
6. What would you expect the graph of $y = T(t)$ look like?

HOMEWORK: Section 1.1: 1 - 121 every other odd.

[1]Does anyone remember what this graph looks like without using desmos?